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Virtually embedded boundary slopes

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Abstract

We show that for certain hyperbolic manifolds all boundary slopes are slopes of π_1-injective immersed surfaces, covered by incompressible embeddings in some finite cover. The manifolds include hyperbolic punctured torus bundles and hyperbolic two-bridge knots. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let M be a manifold with boundary consisting of a single torus, T. A *slope*, α, is an essential simple closed curve in T. The slope is an *embedded boundary slope* if there is an embedded surface in M, whose boundary consists of loops in T parallel to α. The surface must be compact and orientable. It must also be properly embedded, π_1-injective, and not properly homotopic rel boundary to any part of the boundary of M.

We can define boundary slopes for manifolds with more than one torus boundary component in the same way. Then a properly embedded surface will have a boundary slope defined for each boundary component of the manifold it intersects. If the surface does not intersect a particular boundary component, then the slope of the surface on that component is not defined.

If we only require the surface to be an immersion, though still embedded in a neighbourhood of T, then the slope is an *immersed boundary slope*. If the immersion is covered by an embedding into some finite cover of M, then the slope is a *virtually embedded boundary slope*.

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It has been shown that a knot can have only finitely many embedded boundary slopes [5], and many examples of such surfaces have been constructed, for example, [6]. Examples of immersed boundary slopes have been found in the figure eight knot [8,9]. Also Baker and Cooper [2] show that for this knot every slope with even numerator is a virtually embedded boundary slope. It has been shown that a punctured torus bundle may have infinitely many virtually embedded boundary slopes [1], and that there is a manifold with every slope an immersed boundary slope [11].

Note that there will always be one slope on ∂M which is null homologous in $H_1(M;\mathbb{R})$, label this slope l. Choose a slope m, such that $m \cap l$ is a single point. These two curves form a basis for $H_1(\partial M;\mathbb{R})$. If M is a knot space, choose l and m to be the longitude and meridian of the knot. We will label the slope a/b if the slope is made up of a meridians and b longitudes.

In this paper we show:

Theorem 1.1. *If M has hyperbolic interior, and a finite cover $\widetilde{M}$ such that*:

(1) *$\widetilde{M}$ has at least three boundary components.*
(2) *There is a boundary torus $\widetilde{T}$ of $\widetilde{M}$ which is a one-fold covering of ∂M.*
(3) *The projection $\rho : \ker i_* \to H_1(\widetilde{T};\mathbb{R})$ is onto, where $i_* : H_1(\partial\widetilde{M};\mathbb{R}) \to H_1(\widetilde{M};\mathbb{R})$ is the map induced by inclusion, and ρ is the vector space projection $\rho : H_1(\partial\widetilde{M};\mathbb{R}) \to H_1(\widetilde{T};\mathbb{R})$.*

Then every slope of ∂M is a virtually embedded boundary slope.

Note that in condition (3), the map ρ is projection onto $H_1(\widetilde{T};\mathbb{R})$, which is a *subspace* of $H_1(\partial\widetilde{M};\mathbb{R})$, and this map has nothing to do with the covering projection $p : \widetilde{M} \to M$, and in fact is not induced by any continuous map between manifolds.

In [2] it is shown that if there is a surface with boundary slope a/b on $\widetilde{T}$, which is not the fiber of a fibration, then a/b is a virtually embedded boundary slope of $\widetilde{T}$. As $\widetilde{T}$ is a one-fold covering of ∂M, these surfaces project down to virtually embedded surfaces in M with the same boundary slope. We then use the Thurston norm to show that there are surfaces of every boundary slope on $\widetilde{T}$, which are not fibers of fibrations.

By constructing particular covers we then show:

Corollary 1.2. *For hyperbolic punctured torus bundles, every boundary slope is a virtually embedded boundary slope.*

Corollary 1.3. *If K is a knot in $S^2 \times S^1$ such that $M = S^2 \times S^1 - K$ is hyperbolic, and the algebraic intersection number of K with $S^2 \times \{point\}$ is at least three, then every slope of M is an embedded boundary slope.*

Corollary 1.4. *For hyperbolic two bridge knots, every boundary slope is a virtually embedded boundary slope.*

2. General discussion

Proof of Theorem 1.1. Let M be a hyperbolic 3-manifold with boundary consisting of a single torus, and let $\widetilde{M}$ be a finite cover of M. Note that $\widetilde{M}$ will also be hyperbolic and hence irreducible, and that $\partial\widetilde{M}$ will be incompressible in $\widetilde{M}$.

The main result we will use is:

Theorem 2.1 [2, Theorem 1.4]. *Let $\widetilde{M}$ be a compact, connected, orientable, atoroidal and irreducible* 3*-manifold, with boundary a finite number of tori. Suppose that S is a connected, nonseparating, orientable, incompressible surface properly embedded in $\widetilde{M}$, which is not the fiber of a fibration of $\widetilde{M}$. Also suppose that ∂S contains some components with slope α, on a torus, $\widetilde{T}$, in the boundary of $\widetilde{M}$. Then α is a virtually embedded boundary slope.*

Moreover there is a finite cyclic cover $\widetilde{M}_2$, and a compact, connected, orientable, incompressible, boundary-incompressible, surface F, properly embedded in $\widetilde{M}_2$. The boundary of F consists of a nonempty set of essential, parallel curves lying on some component U of $\partial\widetilde{M}_2$ which covers $\widetilde{T}$. Also the covering map is an immersion on F, which is an embedding in a neighbourhood of the boundary, and the boundary of F is mapped to loops parallel to α.

This shows that it suffices to find a connected surface S, in some finite cover $\widetilde{M}$ of M, with the following properties:

(i) S has the required boundary slope on the boundary torus $\widetilde{T}$ of $\widetilde{M}$.

(ii) S is not the fiber of a fibration.

We will need to know when a surface is not a fiber of a fibration. For this we will use the following results about the Thurston norm:

Definition 2.2 [14, Section 1, pp. 103–105]. If S is a connected surface, let $\chi_-(S) = \max\{0, -\chi(S)\}$, where $\chi(S)$ is the Euler characteristic of S. If S has connected components $S_1, \ldots, S_k$, define $\chi_-(S) = \chi_-(S_1) + \cdots + \chi_-(S_k)$.

Define the *Thurston norm* $x(.)$ on $H_2(M, \partial M; \mathbb{Z})$ by

$$x(s) = \min\bigl\{\chi_-(S) \mid S \text{ is an embedded surface representing } s \in H_2(M, \partial M; \mathbb{Z})\bigr\}.$$

This extends to a function on $H_2(M, \partial M; \mathbb{Q})$ by linearity, and then to a function on $H_2(M, \partial M; \mathbb{R})$ by continuity.

Theorem 2.3 [14, Theorem 1, p. 100]. *The function $x(.)$ defined on $H_2(M, \partial M; \mathbb{R})$ is convex and linear on rays through the origin. If every embedded surface representing a nonzero element of $H_2(M, \partial M; \mathbb{R})$ has negative Euler characteristic, then $x(.)$ is a norm. In general $x(.)$ is a pseudonorm vanishing on precisely the subspace spanned by embedded surfaces of nonnegative Euler characteristic.*

Theorem 2.4 [14, Theorem 2, p. 106]. *When $x(.)$ is a norm, the unit ball of $H_2(M, \partial M; \mathbb{R})$ is a polyhedron defined by linear inequalities with integer coefficients, with respect to a basis of primitive elements of $H_2(M, \partial M; \mathbb{Z})$.*

Theorem 2.5 [14, Theorem 3, p. 113]. *If the fiber of M is a surface with negative Euler characteristic, then the ray determined by the homology class of any fiber passes through the interior of a top-dimensional face of the unit sphere.*

First we show that $x(.)$ is a norm on $H_2(\widetilde{M}, \partial\widetilde{M}; \mathbb{R})$.

Lemma 2.6. *If $\widetilde{M}$ satisfies the conditions given above, then the Thurston norm $x(.)$ is a norm on $H_2(\widetilde{M}, \partial\widetilde{M}; \mathbb{R})$.*

Proof. It suffices to show that if $x(s) = 0$, then s is zero in $H_2(\widetilde{M}, \partial\widetilde{M}; \mathbb{R})$.

If $x(s) = 0$, then s is represented by an embedded surface S, whose connected components must all be discs, annuli, spheres, or tori.

An embedded sphere in $\widetilde{M}$ bounds a ball, as $\widetilde{M}$ is irreducible, so all embedded spheres represent trivial homology in $H_2(\widetilde{M}, \partial\widetilde{M}; \mathbb{R})$.

A properly embedded disc D must bound a disc D' in $\partial\widetilde{M}$, as the boundary of $\widetilde{M}$ is incompressible. These two discs bound a ball, as $\widetilde{M}$ is irreducible, so D represents a trivial homology class in $H_2(\widetilde{M}, \partial\widetilde{M}; \mathbb{R})$ as well.

Suppose S has a torus component Y. If Y is π_1-injective then Y must be parallel to a boundary component of $\widetilde{M}$, as $\widetilde{M}$ is hyperbolic. Therefore Y represents trivial homology.

If T is not π_1-injective, then there is a compressing disc D for Y in $\widetilde{M}$. Surger Y along D to produce a sphere, which bounds a ball in $\widetilde{M}$ by irreducibility. So Y is a boundary, as Y bounds the union of this ball and a regular neighbourhood of D, so Y must be trivial in $H_2(\widetilde{M}, \partial\widetilde{M}; \mathbb{R})$.

Suppose a component of S is an annulus, A. If a component of ∂A bounds a disc D in $\partial\widetilde{M}$, then D can be pushed off the boundary of $\widetilde{M}$ so that $A \cup D$ is a properly embedded disc in $\widetilde{M}$. This bounds a region R in $(\widetilde{M}, \partial\widetilde{M})$. So A bounds the union of R and a regular neighbourhood of D in $(\widetilde{M}, \partial\widetilde{M})$, so A is trivial in $H_2(\widetilde{M}, \partial\widetilde{M}; \mathbb{R})$.

If both components of ∂A are contained in the same boundary component Y of $\widetilde{M}$, then the annulus A forms a homotopy in $\widetilde{M}$ between them. As Y is π_1-injective, and

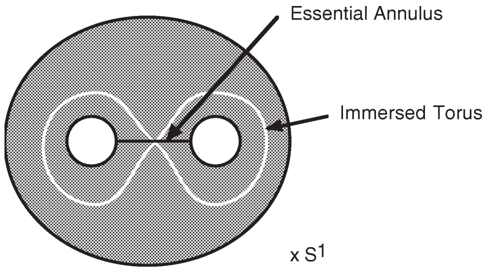

Fig. 1. Cross section of the manifold.

neither curve is trivial, they must bound an annulus A' in Y. Form a new torus Y' from Y by replacing A' with A. The embedded torus Y' in $\widetilde{M}$ is trivial, so A must be trivial in $H_2(\widetilde{M}, \partial\widetilde{M}; \mathbb{R})$ as well.

Suppose the components of ∂A are contained in different boundary components, Y_1 and Y_2, of $\widetilde{M}$. Let N be a regular neighbourhood of $Y_1 \cup A \cup Y_2$. As neither component of ∂A bounds a disc in $\partial\widetilde{M}$, ∂N is an embedded torus in $\widetilde{M}$, so it must be parallel to a boundary component. This means the manifold must be a solid torus with two parallel cores drilled out, as in Fig. 1.

However this manifold is not hyperbolic as it contains an immersed torus which is not boundary parallel. So in fact no such essential annuli can exist.

This shows that if $x(s) = 0$, then $s = 0$ in $H_2(\widetilde{M}, \partial\widetilde{M}; \mathbb{R})$, so $x(.)$ is a norm. □

Suppose $\widetilde{M}$ and $\widetilde{T}$ are as in Theorem 1.1. We can use the Thurston norm to find surfaces which are not fibers, using the relative homology exact sequence, as follows:

$$\begin{tikzcd} H_2(\widetilde{M}, \partial\widetilde{M}; \mathbb{R}) \arrow[r, "\partial_*"] & H_1(\partial\widetilde{M}; \mathbb{R}) \arrow[r, "i_*"] \arrow[d, "\rho"'] & H_1(\widetilde{M}; \mathbb{R}) \\ & H_1(\widetilde{T}; \mathbb{R}) \cong \mathbb{R}^2 & \end{tikzcd}$$

Here ρ is the projection $\rho : H_1(\partial\widetilde{M}; \mathbb{R}) \to H_1(\widetilde{T}; \mathbb{R})$, where $\widetilde{T}$ is the boundary torus which is a one-fold covering of ∂M.

A boundary slope a/b defines a 1-dimensional subspace, which we will call a line, in $H_1(\widetilde{T}; \mathbb{R})$. We say a line has *rational slope* if it contains a nonzero integer homology class. We would like to show that the pre-image of this subspace in $H_2(\widetilde{M}, \partial\widetilde{M}; \mathbb{R})$ contains a line with rational slope, which does not pass through the interior of a top-dimensional face of the unit ball.

To do this, we will use the following result:

Lemma 2.7. *Suppose $\widetilde{M}$ is hyperbolic, with at least three boundary components. If the linear map $\phi = \rho \circ \partial_* : H_2(\widetilde{M}, \partial\widetilde{M}; \mathbb{R}) \to H_1(\widetilde{T}; \mathbb{R})$ is onto, then for any line L in $H_1(\widetilde{T}; \mathbb{R})$, there is a line V in $\phi^{-1}(L)$, such that $\phi(V) = L$, and V does not pass through the interior of a top-dimensional face of the unit ball under the Thurston norm.*

Furthermore, if the line L has rational slope, then V can be chosen to have rational slope as well.

Proof. By Poincaré duality, and the relative homology long exact sequence, we know that the dimension of the kernel of the inclusion map $i_* : H_1(\partial\widetilde{M}; \mathbb{R}) \to H_1(\widetilde{M}; R)$ is half the dimension of $H_1(\partial\widetilde{M}; \mathbb{R})$. The manifold $\widetilde{M}$ has at least three boundary components, so $\dim\ker i_* \geqslant 3$. By the relative exact homology sequence image $\partial_* = \ker i_*$, so $H_2(\widetilde{M}, \partial\widetilde{M}; \mathbb{R}) \cong \mathbb{R}^n$, for some $n \geqslant 3$. Let B be the unit ball of $H_2(\widetilde{M}, \partial\widetilde{M}; \mathbb{R})$, which is a polyhedron by Theorem 2.4.

Let L be a line with rational slope in $H_1(\widetilde{T}; \mathbb{R})$. Then $\phi^{-1}(L)$ is an $(n-1)$-dimensional subspace of $H_2(\widetilde{M}, \partial\widetilde{M}; \mathbb{R})$.

If $\phi^{-1}(L)$ does not intersect the interior of a top-dimensional face of B, then neither does any line in $\phi^{-1}(L)$. So any point of $\phi^{-1}(L) - \ker\phi$ defines a one-dimensional subspace V, such that $\phi(V) = L$, and V does not pass through the interior of a top-dimensional face of B.

If $\phi^{-1}(L)$ does intersect the interior of a top-dimensional face of B, then the intersection has dimension $n - 2 \geqslant 1$, so $\phi^{-1}(L)$ must intersect the boundary of that face in at least $n - 1$ linearly independent points. At least one of these points must be nonzero under ϕ as $\dim\ker\phi = n - 2$. This point defines a line V which does not pass through the interior a top-dimensional face of B. We now need to show V can be chosen to have rational slope.

We have chosen a preferred basis of $H_2(\widetilde{M}, \partial\widetilde{M}; \mathbb{R})$ in which all elements of $H_2(\widetilde{M}, \partial\widetilde{M}; \mathbb{Z})$ are represented by integer multiples of the basis elements. The the faces of B are defined by linear equations and inequalities with integer coefficients. The subspaces $\ker\phi$ and $\phi^{-1}(L)$, are also defined by linear equations with integer coefficients, so if the intersection of $\phi^{-1}(L) - \ker\phi$ with any face of B is nonempty, it will contain a point with rational coefficients. So V can always be chosen to have rational slope. $\square$

We can now prove Theorem 1.1:

Lemma 2.7 shows that for all slopes a/b on $\widetilde{T}$ there is a line V in $H_2(\widetilde{M}, \partial\widetilde{M}; \mathbb{R})$ with rational slope such that $\rho\partial_* V = L$, and V does not pass through the interior of a top-dimensional face of the unit ball. So there is an embedded nonseparating surface S in $\widetilde{M}$, which is not the fiber of a fibration, and which has the required boundary slope on $\widetilde{T}$. Note that we can choose S to be a norm-minimizing surface in its homology class, i.e., $\chi_-(S) = x([S])$, so S will be incompressible.

However, to apply Theorem 2.1 we need a *connected* surface with these properties. Suppose S_1 and S_2 are two surfaces such that $x([S_1 + S_2]) = x([S_1]) + x([S_2])$. Let $[S_1] = q_1 a_1$ and $[S_2] = q_2 a_2$, where $0 < q_i \in \mathbb{Q}$, and $x(a_i) = 1$. Then the intersection of the line through $[S_1 + S_2]$ with the surface of the unit ball is given by

$$\frac{q_1}{q_1 + q_2} a_1 + \frac{q_2}{q_1 + q_2} a_2$$

which lies on the straight line connecting a_1 and a_2. Therefore the line segment between a_1 and a_2 must lie in the surface of the unit ball by convexity, so a_1 and a_2 lie in the same top-dimensional face of B, though not necessarily in its interior.

Suppose S is not connected, with connected components $S_1, \ldots, S_k$. As S is norm-minimizing, $x([S_1 + \cdots + S_k]) = x([S_1]) + \cdots + x([S_k])$, so all the lines defined by the S_i pass through the same top-dimensional face of B. As this face is convex, if any line passes through the interior, then the sum of the homology classes will define a line passing through the interior, so every component of S defines a line which does not pass through a top-dimensional face of B. As S has boundary on $\widetilde{T}$, so does at least one of its connected components. This component is a connected surface with the correct slope on $\widetilde{T}$, which is not a fiber of a fibration.

This completes the proof of Theorem 1.1. $\square$

Remark on condition (2) of Theorem 1.1. What happens if $\widetilde{T}$ is not a degree one cover of ∂M? Let $p:\widetilde{T}\to\partial M$, and let μ and λ be slopes of $\widetilde{T}$ which cover m and l, respectively. If $p(\mu)=q_1m$ and $p(\lambda)=q_2l$, then an embedded surface in $\widetilde{M}$ with slope a/b on T projects down to an immersed surface with slope q_1a/q_2b. However, this will only be an embedding on the boundary if q_1a and q_2b are coprime. For the applications in this paper, we will always be able to choose $\widetilde{T}$ to be a one-fold cover.

Remarks on condition (3) of Theorem 1.1. In order to apply Theorem 1.1, we need to find a boundary component $\widetilde{T}$ of $\widetilde{M}$, such that $\rho:\ker i_*\to H_1(\widetilde{T};\mathbb{R})$ is onto. In this section we investigate what happens when this projection map is not onto.

Let $M(r)$ denote the manifold obtained by Dehn filling M with slope r. Let $\widetilde{M}(r)$ be the manifold obtained by Dehn filling $\widetilde{M}$ such that the filling curve on each boundary component of $\widetilde{M}$ covers the filling curve with slope r on ∂M.

Lemma 2.8. *Suppose $\widetilde{M}$ has at least two boundary components. If the projection $\rho:\ker i_*\to H_1(\widetilde{T};\mathbb{R})$ is not onto, then* $\dim H_2(\widetilde{M}(0))\geqslant 2$.

Proof. Suppose ρ is not onto.

Let $\triangle$ be the *intersection form* on $H_1(\partial\widetilde{M};\mathbb{R})$. The value of $\triangle(\alpha,\beta)$ is defined to be the algebraic intersection number of representatives of the homology classes of α and β in general position. The form $\triangle$ is a skew-symmetric bilinear form on $H_1(\partial\widetilde{M};\mathbb{R})$, which is nonsingular on $H_1(\partial\widetilde{M};\mathbb{R})$, and $\triangle\equiv 0$ on $\ker i_*$.

Note that l is null homologous in $H_1(M;\mathbb{R})$, so there is a surface S in M with boundary parallel to l. There is a pre-image of this surface, $\widetilde{S}$ in $\widetilde{M}$ with boundary on $\widetilde{T}$. Take the surface $\widetilde{S}$ to be the entire pre-image of S, so $\widetilde{S}$ need not be connected, but it does intersect every boundary component of $\widetilde{M}$. The boundary of $\widetilde{S}$ is in $\ker i_*$, so $\rho([\partial\widetilde{S}])$ is a nonzero multiple of l. Therefore if ρ is not onto, then its image must be one-dimensional, generated by l. Let λ be the slope on $\widetilde{T}$ which covers l. Every element of $\ker i_*\cap H_1(\widetilde{T};\mathbb{R})$ must be represented by slopes on $\widetilde{T}$ parallel to λ, as the map induced by the covering map $p_*:H_1(\widetilde{T};\mathbb{R})\to H_1(\partial M;\mathbb{R})$ is injective. So $\triangle(\lambda,\alpha)=0$ for all $\alpha\in\ker i_*$, as every $\alpha\in\ker i_*$ is represented by some surface with boundary parallel to λ on $\widetilde{T}$, or else with no boundary at all on $\widetilde{T}$. The form $\triangle$ is nonsingular, and $\ker i_*$ is a maximal subspace on which it vanishes. So as $\triangle(\lambda,\alpha)=0$ for all $\alpha\in\ker i_*$ this means that $\lambda\in\ker i_*$. Therefore there is a surface S' in $\widetilde{M}$, such that $\partial S'$ is a multiple of λ on $\widetilde{T}$, and the homology of $\partial S'$ is zero on all other boundary components of $\partial\widetilde{M}$. So we can choose this surface S' to have boundary only on $\widetilde{T}$.

Dehn fill all boundary components of M and $\widetilde{M}$ with slope 0, to produce closed manifolds $M(0)$ and $\widetilde{M}(0)$, so that $\widetilde{M}(0)$ is a branched covering of $M(0)$.

Now consider the surfaces formed from $\widetilde{S}$ and S' by capping off their boundaries with meridianal discs of the solid tori filling the boundary components of $\widetilde{M}$. These two surfaces are nonzero elements of $H_2(\widetilde{M}(0);\mathbb{R})$. The cover of a meridian of ∂M on any component of $\partial\widetilde{M}$ is a nonzero homology class in $\widetilde{M}(0)$. As there are at least two components of $\partial\widetilde{M}$, there is a cover of the meridian which intersects $\widetilde{S}$ but not S', so these two surfaces represent linearly independent elements of $H_2(\widetilde{M}(0);\mathbb{R})$. Therefore $\dim H_2(\widetilde{M}(0);\mathbb{R})\geqslant 2$. □

Lemma 2.8 shows that for a boundary torus $\widetilde{T}$ in $\partial\widetilde{M}$, then if the map $\rho : \ker i_* \to H_1(\widetilde{T}; \mathbb{R})$ is not onto, then there is a surface S' with boundary consisting only of parallel copies of λ, the slope on $\widetilde{T}$ which covers l.

Linking numbers, $\mathrm{lk}(\alpha, \beta) \in \mathbb{Q}$, are defined between disjoint simple closed curves which represent elements in the torsion subgroup of $H_1(\widetilde{M}(\infty); \mathbb{Z})$, i.e., $[\alpha] = [\beta] = 0$ in $H_1(\widetilde{M}(\infty); \mathbb{R})$. The manifold $\widetilde{M}(\infty)$ is a branched cover of $M(\infty)$, with branch set the core K of the filling torus of ∂M. For certain branched covers, all of the components of $\widetilde{K}$, the cores of the solid tori filling $\partial\widetilde{M}$, are in the torsion subgroup of $H_1(\widetilde{M}(\infty); \mathbb{R})$, so linking numbers are defined between them. When this happens, the surface S', considered as a surface in $\widetilde{M}(\infty)$, is a surface whose boundary is homologous to some nonzero multiple of the core of $\widetilde{T}$. The surface S' does not intersect any of the other boundary components of $\partial\widetilde{M}$, so it does not intersect any of the cores of the other tori filling them. Therefore the surface S' shows that the core of T must have linking number zero with the cores of the other filling tori. This gives the following result:

Lemma 2.9. *If the projection $\rho : \ker i_* \to H_1(\widetilde{T}; \mathbb{R})$ is not onto, and linking numbers are defined between the cores of the filling tori in $\widetilde{M}(\infty)$, then the core of the torus filling $\widetilde{T}$ has linking number zero with each of the cores of the other filling tori.*

3. Hyperbolic punctured torus bundles

We need to show that there is a cover of a hyperbolic punctured torus bundle satisfying the conditions of Theorem 1.1.

In this section, M will denote the punctured torus bundle, $M = F \times I/(x, 0) \sim (f(x), 1)$, where F is a punctured torus, and $f : F \to F$ is a homeomorphism of the punctured torus to itself.

Pick a basepoint $x_0 \in \partial F$. The loops labelled x and y give a basis for $H_1(F; \mathbb{Z})$, and the loops l and $m = x_0 \times I$ are a basis for $H_1(\partial F; \mathbb{Z})$. This is illustrated in Fig. 2.

Each homeomorphism induces an automorphism, f_* of $H_1(F; \mathbb{Z}) \cong \mathbb{Z} \times \mathbb{Z}$. So $f_* \in SL_2(\mathbb{Z})$. If $|\mathrm{trace}(f_*)| > 2$, then the homeomorphism is said to be *pseudo-Anosov*, and the resulting manifold is hyperbolic [10,12]. If $\mathrm{trace}(f_*) = 3$, then the manifold is the figure eight knot exterior and the homology basis can be chosen so that the loops l and m are the longitude and meridian of the figure eight knot. Note that a punctured torus bundle is irreducible and has incompressible boundary.

We will construct the covering space in the following way:

The fundamental group of F is the free group on two generators, $\langle x, y \mid \rangle$. Let $\phi : \pi_1 F \to \mathbb{Z}_3 \oplus \mathbb{Z}_3$ be the homomorphism that sends $x \mapsto \binom{1}{0}$ and $y \mapsto \binom{0}{1}$, where $\mathbb{Z}_3$ is the cyclic group of order 3. Then $\ker\phi$ defines a 9-fold regular covering space $\widetilde{F}$ of F. The subgroup $\ker\phi$ is a characteristic subgroup of $\pi_1 F$, so $f_*(p_*\pi_1\widetilde{F}) = p_*\pi_1\widetilde{F}$, and there is a homeomorphism $\tilde{f} : \widetilde{F} \to \widetilde{F}$ which covers f.

The covering translations of $\widetilde{F}$ form a group isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$. Choose a basepoint $\tilde{x}_0$ for $\widetilde{F}$ on a boundary component such that $p(\tilde{x}_0) = x_0$, and label this boundary

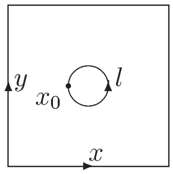

Fig. 2. A punctured torus.

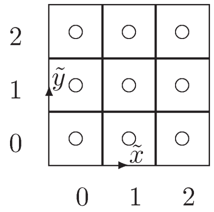

Fig. 3. The 3×3-fold cover.

component $L\binom{0}{0}$. Label each other boundary component by the covering translation which maps $L\binom{0}{0}$ onto it. This labeling is illustrated in Fig. 3. It is easy to check that with respect to the standard homology basis $\tilde{f}$ permutes the boundary components, as labelled, in the same way that the matrix of f_* permutes the elements of $\mathbb{Z}_3 \times \mathbb{Z}_3$ (the 2×2 matrix of integers f_* acts on $\mathbb{Z}_3 \times \mathbb{Z}_3$ in the obvious way). As the monodromy homeomorphism $f: F \to F$ is covered by a homeomorphism $\tilde{f}: \widetilde{F} \to \widetilde{F}$, so $\widetilde{M} = \widetilde{F} \times I/(x,0) \sim (\tilde{f}(x), 1)$ is a covering space for M. Note that $\widetilde{M}$ is an irregular covering of M for the punctured torus bundles we are considering, even though $\widetilde{F}$ is a regular covering of F.

Note also that as f is pseudo-Anosov, $\tilde{f}$ will be pseudo-Anosov.

The top of the cylinder formed by the boundary component $L\binom{a}{b} \times I$ of $\widetilde{F} \times I$, is identified with the boundary component of F in the position given by $f_*\binom{a}{b}$. Think of the linear map f_* acting on the two-dimensional vector space $\mathbb{Z}_3 \times \mathbb{Z}_3$ as a permutation, then the number of boundary components of $\widetilde{M}$ is equal to the number of cycles of f_*. Note that as $\binom{0}{0}$ always gets mapped to itself, the permutation will always have at least one 1-cycle. The boundary torus formed by this boundary component of $\widetilde{F}$ will be chosen to be the boundary torus $\widetilde{T}$, which is a one-fold covering of ∂M.

The following table shows the cycles of f_*. All conjugacy classes of $PSL_2(\mathbb{Z}_3)$ are listed, labelled by a single element chosen from each conjugacy class:

Conjugacy class of f_*	Length of cycles in $\mathbb{Z}_3 \times \mathbb{Z}_3$
$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$1, 1, 1, 1, 1, 1, 1, 1, 1$
$\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}1&2\\0&1\end{pmatrix}$	$1, 1, 1, 3, 3$
$\begin{pmatrix}2&0\\0&2\end{pmatrix}$	$1, 2, 2, 2, 2$
$\begin{pmatrix}2&1\\0&2\end{pmatrix}\begin{pmatrix}2&2\\0&2\end{pmatrix}$	$1, 2, 6$
$\begin{pmatrix}0&2\\1&0\end{pmatrix}$	$1, 4, 4$

So $\widetilde{M}$ always has at least three boundary components.

We will use Lemma 2.8 to show that the projection $\rho: \ker i_* \to H_1(\widetilde{T}; \mathbb{R})$ is onto.

Consider Dehn filling all boundary components of M and $\widetilde{M}$ with slope 0, to produce closed manifolds $M(0)$ and $\widetilde{M}(0)$. For the particular covering we have chosen, all pre-images of l cover l one-to-one, so the branching index of each Dehn filling is one. Therefore the manifold $\widetilde{M}(0)$ covers $M(0)$, which is a torus bundle with Anosov monodromy, so $\widetilde{M}(0)$ must also be a torus bundle with Anosov monodromy. In particular, this implies $H_2(\widetilde{M}(0);\mathbb{R}) \cong \mathbb{R}$. However this contradicts Lemma 2.8, so in fact ρ must be onto.

So $\widetilde{M}$ satisfies the conditions of Theorem 1.1. Therefore for hyperbolic punctured torus bundles, every slope is a virtually embedded boundary slope. This proves Corollary 1.2. □

4. Knots in $S^2 \times S^1$

We can use Lemma 2.8 whenever we know that $H_2(\widetilde{M}(0);\mathbb{R}) \cong \mathbb{R}$. In particular, this happens for all finite covers of $S^2 \times S^1$.

Let K be a knot in $S^2 \times S^1$ such that the algebraic intersection number of K with $S^2 \times \{\text{point}\}$, $|\triangle(K, S^2 \times \{\text{pt}\})| = n \geqslant 3$, and assume further that $M = S^2 \times S^1 - K$ is hyperbolic.

Take the n-fold cover of $S^2 \times S^1$. The knot K lifts to a link with n components, each of which is a degree one cover of K. As this covering of M does not unwrap ∂M in the direction of the longitude, the covering $p : \widetilde{M} \to M$ extends to a covering $\bar{p} : \widetilde{M}(0) \to M(0)$.

Suppose the projection $\rho : \ker i_* \to H_1(T;\mathbb{R})$ is not onto, then by Lemma 2.8,

$$\dim H_2\big(\widetilde{M}(0);\mathbb{R}\big) \geqslant 2.$$

But $\widetilde{M}(0)$ covers $M(0) = S^2 \times S^1$, so $\widetilde{M}(0)$ must also be $S^2 \times S^1$. But then $H_2(\widetilde{M}(0);\mathbb{R}) \cong \mathbb{R}$, which gives a contradiction.

Therefore, for these manifolds, every slope is a virtually embedded boundary slope. This proves Corollary 1.3. □

5. Two-bridge knots

In this section, the manifold M will always be a hyperbolic knot space, i.e., the complement of the interior of a regular neighbourhood of the knot K in S^3. We write $b(\alpha, \beta)$ to denote the two-bridge knot which gives the lens space $L(\alpha, \beta)$, when used as the branch set for a two-fold branched cover of S^3. If $b(\alpha, \beta)$ is a knot, then α is odd. The only closed incompressible surfaces in two bridge knots are boundary parallel tori [7], so all two bridge knots are hyperbolic, except those that are torus knots [13].

In general, the cores of the filling tori of $\widetilde{M}(\infty)$, need not be null-homologous in $H_1(\widetilde{M};\mathbb{R})$, so linking numbers need not exist between them. However linking numbers do exist for particular classes of branched covers, corresponding to dihedral representations of knot groups, which have been extensively studied. For full details of all the results used about dihedral coverings see [4, Chapter 14].

Suppose we have a representation $\phi : \pi_1 M \to D_{2n} = \mathbb{Z}_2 \ltimes \mathbb{Z}_n$, with n odd. The fundamental group of M can be written as $\mathbb{Z} \ltimes G$, where $\mathbb{Z}$ is generated by a meridian of the knot, and G is the commutator subgroup of $\pi_1 M$. Therefore m must get mapped onto a reflection in D_{2n}. The longitude l is in the commutator subgroup of $\pi_1 M$, so $\phi(l)$ is in the $\mathbb{Z}_n$ subgroup of rotations. But l and m commute, so $\phi(l) = 1$. So there is a regular $2n$-fold covering space M_{2n} corresponding to $\ker\phi$, which has n boundary components, each of which is a two-fold cover of ∂M. Let A be the $\mathbb{Z}_2$ subgroup generated by $\phi(m)$. Then $\phi^{-1}(A)$ generates an irregular n-fold covering M_n of M.

We need to know how many boundary components the cover M_n has. Choose a point x in ∂M. The group $\pi_1 M$ acts on $p^{-1}(x)$ on the right by path lifting, i.e., if $\tilde{x} \in p^{-1}(x)$, and α is a loop based at x in M, then $\tilde{x}\alpha = \tilde{\alpha}(1)$, where $\tilde{\alpha}$ is the unique lift of α such that $\tilde{\alpha}(0) = \tilde{x}$. As a right $\pi_1 M$ space, $p^{-1}(x)$ is isomorphic to the space of right cosets of $p_*\pi_1 M_n$ in $\pi_1 M$. This in turn is isomorphic to the space of right cosets of A in D_{2n}, as an element $[\alpha] \in \pi_1 M$ acts on A by right multiplication by $\phi([\alpha])$. Choose two elements $x_1,\ x_2$ of $p^{-1}(x)$ and label them by the right A cosets, $Aa_1,\ Aa_2$, they correspond to. The two elements $x_1,\ x_2$ of $p^{-1}(x)$ lie in the same boundary component of ∂M_n, if and only if there is a path α in ∂M_n connecting them. This path projects down to a loop $p(\alpha)$ in ∂M, which gets mapped into A by ϕ, as $\phi(\pi_1 \partial M) = A$. So $Aa_1\phi([p(\alpha)]) = Aa_2$. This element $\phi([p(\alpha)])$ of A must map one coset to the other, and if any element of A does so, then there is a corresponding path in ∂M_n connecting the two points. Therefore two elements of $p^{-1}(x)$ lie in the same boundary component if and only if their corresponding cosets lie in the same (A, A)-double coset of D_{2n}. Furthermore the order of the covering of each boundary component is given by the number of cosets of A in each double coset.

A simple calculation shows that the number of (A, A)-double cosets in D_{2n} is $(n+1)/2$, all of which contain two cosets of A, except A itself. Therefore the cover M_n has $(n+1)/2$ boundary components, one of which is a one-fold cover. Choose this boundary component to be $\widetilde{T}$, which is covered two-to-one by a single boundary component of M_{2n}. There are n cosets of A in D_{2n}, so M_{2n} has n boundary components. Therefore all the boundary components of M_n except $\widetilde{T}$ are covered by two boundary components of M_{2n}.

Theorem 5.1 [4, Theorem 14.8]. *There is a surjective homomorphism $\phi : \pi_1 M \to D_{2p}$, if and only if the prime p divides the order of $H_1(C_2)$, where C_2 is the two-fold branched cover of S^3, branched over the knot K. If p does not divide the second torsion coefficient of $H_1(C_2)$, then all such representations are equivalent.*

Theorem 5.2 [4, Proposition 14.16]. *If there is exactly one class of equivalent dihedral homomorphisms $\phi : \pi_1 M \to D_{2p}$, then linking numbers are defined between the core curves of M_{2p}, and also M_p.*

These linking numbers, when they exist, have been computed for many of the knots in the knot tables. However, in the case of two bridge knots, they always exist.

Theorem 5.3 [3]. *Let M be the knot space of the two bridge knot $b(\alpha, \beta)$. There is a dihedral representation $\phi : \pi_1 M \to D_{2\alpha}$. Linking numbers are defined in the branched covers corresponding to* $\ker\phi$, *and $\phi^{-1}(A)$, for any reflection subgroup $A \cong \mathbb{Z}_2$, even if α is not prime. The linking numbers in $M_{2\alpha}$ are ± 1 for all pairs of core curves in $M_{2\alpha}$.*

The covers $M_{2\alpha}$ and M_α both give rise to branched covers of S^3, namely $M_{2\alpha}(\infty)$ and $M_\alpha(\infty)$, in the notation of Section 2. Note that $M_{2\alpha}(\infty)$ is a two-fold branched cover of $M_\alpha(\infty)$, branched over t, the core of $\widetilde{T}$. Let c be some other core curve in $M_\alpha(\infty)$, which will have two pre-images $\tilde{c}_1$ and $\tilde{c}_2$ in $M_{2\alpha}$. If $\mathrm{lk}(t, c) = 0$, then $\mathrm{lk}(\tilde{t}, \tilde{c}_i)$ will also be zero for each i, but by Theorem 5.3, $\mathrm{lk}(\tilde{t}, \tilde{c}_i) = \pm 1$ for both pre-images of c. Therefore $\mathrm{lk}(t, c) \neq 0$ for all core components $c \neq t$, so by Lemma 2.9, $\rho : \ker i_* \to H_1(\widetilde{T}; \mathbb{R})$ is onto. If $p \geqslant 5$, then M_α has at least three boundary components, so by Theorem 1.1, every boundary slope is a virtually embedded boundary slope. The only two-bridge knot with $\alpha < 5$ is the trefoil, which is not hyperbolic, so this proves Corollary 1.4.

Table III in [4] lists linking invariants of knots, from which the linking numbers can easily be computed. This shows that many of the hyperbolic knots in the tables have virtually immersed boundary slopes of every slope.

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